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### Metric for object association in the Keplerian orbits space

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#### Abstract

Orbit determination of space debris objects in the context of space surveillance is often performed by associating several sequences of observations related to the same object. If preliminary orbits can be computed from the single sequences, there is a need for an association criterion to evaluate whether they belong to a unique object. The definition of a distance between orbits allows for a threshold criterion for which two orbits are considered associated. Several definitions of distance exist that satisfy the metric space axioms. These depend on the choice of the parameters to describe the orbit. For Keplerian orbits one possible definition formulates the distance as a function of the traditional Keplerian orbital elements. The appropriate metric changes according to the degrees of freedom considered in the problem.

In this paper we propose a new metric that includes the orbit anomaly in addition to the other orbital parameters which characterize the orbital plane and shape. Contrary to models adopted in the asteroid research, in the case of space debris and satellite orbits, where the revolution periods are much smaller, considering the position of the object along the orbit can significantly improve the orbit characterization and therefore the orbit association. Additionally, in our approach the proposed metric is scaled according to the obtained orbit covariances. Applications of the distance definition to several association examples are shown.

#### 1 Introduction

The regular observation of space debris is important to characterize the near Earth space environment and collect data useful for space surveillance purposes. The observations are usually processed to orbits which are assigned to space objects in a catalogue. Often it is necessary to associate groups of observations belonging to the same object to calculate its orbit. Refer e.g. to [RD-1][RD-2] for details about some association methods. In case preliminary orbits can be computed from these single groups (also called tracklets), an association criterion based on these orbits is necessary to evaluate whether the observations are related to the same object [RD-6]. The definition of a distance between orbits allows for a threshold criterion to make the association. The association is then successful if the distance is smaller than a given threshold. Often the definition of Mahalanobis distance is used as a measure of the

goodness of the association. The orbits are transformed to state vectors and the Mahalanobis distance is evaluated in Cartesian coordinates. The orbit uncertainty is described according to a Gaussian distribution and mostly the Gaussian model is good enough to describe it, but depending on the coordinate system the inadequacy can be accentuated (see e.g. [RD-5]). An appropriate coordinate system can be found where the Gaussian assumption well approximates the actual distribution. In curvilinear coordinates [RD-4][RD-6] the orbital uncertainty distribution can be better described, since the coordinates take into account the real curved trajectory of the object.

Other definitions of distance are formulated in the space of the orbits, namely considering the orbital elements as the components of the vector space. Several studies have investigated the structure of the space of Keplerian orbits. Moser [RD-8] first studied the space of constant energy surfaces for bounded Keplerian orbits. He showed that this space is topologically equivalent to the Cartesian product of two spheres. An explicit formula for the geodesic distance between points in this space was derived in [RD-3]. In [RD-9] different metrics in the orbits space are proposed.

In this work an extended definition of distance is proposed which adds the anomaly in the definition in [RD-3], the latter being based only on five Keplerian orbital elements. In general, the existing definitions of orbit distance come from the asteroid research, characterized by the long orbital period of the observed asteroids. There, especially the shape and the orientation of the orbital plane are relevant for a comparison, less the position of the object along the orbit (given by the anomaly) which is difficult to be predicted. However, in the case of space debris and satellite orbits, different possible observation conditions (e.g. re-observation after a fraction of the orbital period) allow for a better prediction of the position and the addition of the anomaly in the distance formulation can improve the orbit association.

In addition to considering the anomaly, as an alternative to the definition in [RD-3] also one of the metrics proposed in [RD-9] is analysed. Both formulations for the distance need to be modified in order to be expressed as a Mahalanobis distance.

#### 2 Distance between Keplerian orbits

In [RD-3] the space of bounded Keplerian orbits of fixed

energy is described using the topology  $\mathbb{V}(E) \sim S^2 \times S^2$ , the Cartesian product of two spheres. This topology is extended through the semimajor axis to the cone  $\mathbb{K}(S^2 \times S^2)$ . The formula to compute the geodesic distance between points in this space is:

$$d = \sqrt{2(a_1^2 + a_2^2 - 2a_1a_2 \cos \psi)}, \quad (1)$$

$$\text{where } \psi = \sqrt{\frac{\arccos^2(\vec{\eta}_1 \cdot \vec{\eta}_2) + \arccos^2(\vec{\xi}_1 \cdot \vec{\xi}_2)}{2}} \quad (2)$$

$$\text{and } \vec{\eta} = \vec{e} + \vec{h}, \quad \vec{\xi} = \vec{e} - \vec{h}. \quad (3)$$

Here is  $\vec{e}$  the eccentricity vector and  $\vec{h}$  a normalized angular momentum vector  $\vec{h} = \frac{\vec{H}}{\sqrt{\mu a}}$  given the semimajor axis  $a$  and the gravitational parameter  $\mu$ . The related Riemannian metric is induced by the Euclidean metric on  $\mathbb{R}^6$ . In the article it is mentioned that the geodesic distance on  $\mathbb{K}(S^1 \times S^1)$  can be generalized to a manifold with  $n$  spheres  $\mathbb{K}(S^1 \times \dots \times S^1) \subset \mathbb{R}^{2n}$ . The definition in (2) is then replaced by a general expression which contains not only specifically the angle differences on the sphere for  $\vec{\eta}$  and  $\vec{\xi}$ , but additional angles for any additional sphere:

$$\psi = \sqrt{\frac{\sum_{i=1}^n \theta_i^2}{n}}. \quad (4)$$

### 3 Distance with mean anomaly

We want to extend the distance between two orbits (1) to all 6 orbital parameters including the orbit anomaly. We can describe the problem with a topology  $\mathbb{H} = \mathbb{K}(S^2 \times S^2 \times S^1) \subset \mathbb{R}^6$  to be able to make use of eq. (4). The sphere  $S^1$  is per definition a subspace of  $\mathbb{H}$  and should be homeomorphic to the set of object positions along the orbit. This can be parameterized with an angle  $\varphi$  that describes the portion of the orbit covered by the object from a given start point. Possible known parameterizations in  $\varphi$  are true, mean, and eccentric anomaly. However, the description of true and eccentric anomaly depends on the eccentricity  $e$  of the orbit. If we consider the sets  $D_1 = \{\varphi: \varphi \in [0, 2\pi)\}$  and  $D_2 = \{e: e \in [0, 1]\}$  we see that there is no bijection  $D_1 \times D_2 \rightarrow S^1$ . Furthermore,  $D_2$  is already related to  $S^2 \subset \mathbb{H}$  through a homeomorphism as previously explained in the construction  $\mathbb{V}(E) \sim S^2 \times S^2$ .

Thus let us examine the case of the mean anomaly. First we consider the mapping to  $S^1$  as  $f: [0, 2\pi) \rightarrow S^1$  with

$$f(\varphi) = (\cos \varphi, \sin \varphi) \quad (5)$$

where  $\varphi \in [0, 2\pi)$  and  $(x, y) \in S^1 \subset \mathbb{R}^2$ .

The mean anomaly at an arbitrary time  $t$  is defined as

$$M(t) = n(t - \tau) \quad (6)$$

where  $\tau$  is the time of pericenter passage and  $n = \frac{2\pi}{T}$  is the mean angular motion.

There is a mapping to the set of mean anomalies  $D = \{M(t), t \in \mathbb{R}\}$  defined as  $g: [0, 2\pi) \rightarrow D$  with

$$g(\varphi) = n\left(\frac{\varphi}{n} - \tau\right) \quad (7)$$

where  $\varphi \in [0, 2\pi)$ .

To account for the compactness in  $S^1$ , we have to extend the half-open interval  $[0, 2\pi)$  in  $f$  and  $g$  to the compact factor group  $\mathbb{R} / (2\pi\mathbb{Z})$  and restrict the mean anomaly to  $D / (2\pi\mathbb{Z})$ . Then the function

$$h = g^{-1} \circ f: D / (2\pi\mathbb{Z}) \rightarrow S^1 \quad (8)$$

is a homeomorphism. This justifies the topology  $\mathbb{H}$  for the extension of the distance definition using also the mean anomaly. Consequently in eq. (4) an additional angle difference  $\theta_i$ , the difference in mean anomaly, is introduced.

### 4 Mahalanobis distance

Eq. (1) in  $\mathbb{R}^2$  is simply proportional to the distance  $d$  in the triangle shown in Figure 1 calculated with the law of cosines. If  $\psi$  is small  $d$  can be approximated by:

$$d \approx \sqrt{(a_2 - a_1)^2 + (a_1\psi)^2} \quad (9)$$

For the generalization in  $\mathbb{R}^6$  the eq. (4) for the angle  $\psi$  has to be used instead.

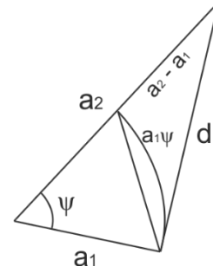


Figure 1: Triangle showing the relation between  $a_1$ ,  $a_2$ ,  $\psi$ , and  $d$  in  $\mathbb{R}^2$ .

The sum of squares appearing in (4) and (9) suggests the possibility to scale the summands according to their uncertainty as in the Mahalanobis distance. We define the vector:

$$\vec{z} = \begin{pmatrix} \frac{\sqrt{3}}{a_1} (a_2 - a_1) \\ \theta_1 \\ \theta_2 \\ M_2 - M_1 \end{pmatrix}, \quad (10)$$

where  $\theta_1 = \arccos(\vec{\eta}_1 \cdot \vec{\eta}_2)$  and  $\theta_2 = \arccos(\vec{\xi}_1 \cdot \vec{\xi}_2)$ .

To find the covariance matrix  $C_{\vec{z}} = Cov(\vec{z})$  we calculate the matrix of the partial derivatives with the components

$$T_{ij} = \frac{\partial \vec{z}_i}{\partial p_j}, \quad (11)$$

where  $\vec{p} = (a_1, \dots, M_1, a_2, \dots, M_2)$  contains the orbital elements of the two orbits. Setting  $C_{\vec{p}} = Cov(\vec{p})$  we

have:

$$C_{\vec{z}} = TC_{\vec{p}}T^T. \quad (12)$$

The Mahalanobis distance  $d_M$  is obtained from:

$$d_M = \sqrt{\vec{z}^T C_{\vec{z}}^{-1} \vec{z}} \quad (13)$$

## 5 Equivalent metric

In [RD-9] an alternative metric is proposed which is also based on the angular momentum  $\vec{K}$  and eccentricity  $\vec{e}$ . The vectors  $\vec{u}$  and  $\vec{v}$  parallel to  $\vec{K}$  and  $\vec{e}$  with the following norm are considered:

$$|\vec{u}| = \sqrt{p}, \quad |\vec{v}| = e\sqrt{p}, \quad (14)$$

where  $p$  is the semilatus rectum.

The metric is defined in the space  $\mathbb{H}$  of all non-rectilinear Keplerian orbits described by five Keplerian elements without the anomaly. The distance in the space  $\mathbb{H}$  is then expressed by the Euclidean distance in the ambient space  $\mathbb{R}^6$ :

$$d = \sqrt{\frac{(\vec{u}_1 - \vec{u}_2)^2 + (\vec{v}_1 - \vec{v}_2)^2}{L}} \quad (15)$$

where  $L$  is an arbitrary factor which can be set to  $L = 1$ . The distance as a function of the orbital elements looks like

$$d^2 = (1 + e_1^2)p_1 + (1 + e_2^2)p_2 - \frac{2\sqrt{p_1 p_2}(\cos I + e_1 e_2 \cos P)}{L} \quad (16)$$

where

$$\cos I = c_1 c_2 + s_1 s_2 \cos \Delta \quad (17)$$

and

$$\cos P = s_1 s_2 \sin \omega_1 \sin \omega_2 + (\cos \omega_1 \cos \omega_2 + c_2 \sin \omega_1 \sin \omega_1) \cos \Delta + (c_2 \cos \omega_1 \sin \omega_2 - c_1 \sin \omega_1 \cos \omega_2) \sin \Delta \quad (18)$$

with

$$c = \cos i, \quad s = \sin i, \quad \Delta = \Omega_1 - \Omega_2. \quad (19)$$

This metric is topological equivalent to the one in eq. (1). According to the authors the Riemannian metric in  $\mathbb{H}$  proposed in [RD-3] possibly describes the diversity of orbits better than the Euclidean one used in eq. (15) but it is more complicated. Even if the two metrics are topological equivalent the obtained distances have different scaling factors.

Let us consider the case of two equal orbits except for the semimajor axis. Then the distance calculated with eq. (1) is

$$d = \sqrt{2(a_1^2 + a_2^2 - 2a_1 a_2)} = \sqrt{2} |a_1 - a_2|, \quad (20)$$

whereas eq. (16) gives

$$d = \sqrt{(1 + e^2)(p_1 + p_2 - 2\sqrt{p_1 p_2})} = \sqrt{1 + e^2} |\sqrt{p_1} - \sqrt{p_2}| \quad (21)$$

The proportional terms  $\sqrt{2}$  and  $\sqrt{1 + e^2}$  occur also with orbits varying in the other orbital elements and as such can be considered overall scaling factors, which are eliminated through the Mahalanobis distance normalization. The  $\sqrt{p}$  terms seem to replace the semimajor axis  $a$  in eq. (20). One possible way to extend eq. (15) to all six Keplerian elements, including the anomaly, is the following:

$$d = \sqrt{(\vec{u}_1 - \vec{u}_2)^2 + (\vec{v}_1 - \vec{v}_2)^2 + (\vec{w}_1 - \vec{w}_2)^2}, \quad (22)$$

where  $\vec{w}_1$  and  $\vec{w}_2$  are vectors such that

$$\angle(\vec{w}_1, \vec{w}_2) = |M_1 - M_2|, \quad |\vec{w}_1| = \sqrt{p_1}, \quad |\vec{w}_2| = \sqrt{p_2}. \quad (23)$$

Eq. (22) already has a structure suitable for scaling according to the Mahalanobis distance. Here we define vectors of the type

$$\vec{z} = \begin{pmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{pmatrix} \quad (24)$$

and we calculate the distance

$$d_M = \sqrt{(\vec{z}_1 - \vec{z}_2)^T C_{\vec{z}_1 - \vec{z}_2}^{-1} (\vec{z}_1 - \vec{z}_2)} \quad (25)$$

where  $C_{\vec{z}_1 - \vec{z}_2} = Cov(\vec{z}_1) + Cov(\vec{z}_2)$  is determined similarly as in the eqs. (11) and (12).

## 6 Simulations

Preliminary simulations were conducted to evaluate the metric in  $\mathbb{H} = \mathbb{K}(S^2 \times S^2 \times S^1)$  and the Mahalanobis distance described in eq. (13). Radar measurements of LEO objects on almost circular orbits (eccentricity  $< 0.01$ ) at altitudes around 1000 km and 800 km were simulated. The objects from the Space-Track TLE catalogue are observed during one night from a station at 40° latitude. Table 1 shows the values used for the simulation.

Radar pointing	Az. 180°, El. 60°
FoR	Az. 120°, El. 20°
Error ( $\sigma$ ) in range	5 m
Error ( $\sigma$ ) in angle	15'
Interval betw. obs.	10 s

Table 1. Values for the simulated radar observations for radar tracklets association.

After excluding from the total amount of detections the too short tracklets and those where the initial orbit

determination fails, a net number of 191 and 769 tracklet pairs remains for the simulations at 1000 km and 800 km, respectively.

The tracklet association procedure follows the scheme described in [RD-7] but it does not filter the false associations according to an RMS threshold in the orbit improvement in order to see the efficacy of the distance threshold. The Mahalanobis distance in curvilinear coordinate [RD-4] and in the orbits space  $\mathbb{H}$  were calculated and an acceptance threshold of 10 was set for both distances. In these preliminary tests with nearly circular orbits the argument of latitude instead of the mean anomaly was used in the distance definition.

Table 2 shows true and false associations using the Mahalanobis distance in curvilinear coordinates and in the orbit elements space. The results obtained with the orbit elements distance are slightly better than with the distance in curvilinear coordinates.

	Curv. coord.	Orbit elements
1000 km	179 / 71	186 / 58
800 km	728 / 4136	744 / 4107

Table 2. True / false positives using the Mahalanobis distance in curvilinear coordinates and in the orbit elements space.

## 7 Conclusions

In the classical orbit association problem the alternative to the commonly used Mahalanobis distance in Cartesian or curvilinear coordinates is a definition of distance in the space of orbital elements. This space can be described by a different topology and the geodesic distance between two points can be calculated in the defined manifold.

In this paper we showed that the distance between Keplerian orbits considering five orbit elements can be extended to include the difference in the mean anomaly. Also, the equation to compute the orbits distance is suitable to be expressed in the form of a Mahalanobis distance scaled according to the covariance of the orbital elements.

Further analysis showed that an alternative definition of the orbits distance can be formulated using a Euclidean instead of a Riemannian metric. This alternative distance is even more suitable than the previous one to be expressed as a Mahalanobis distance because it consists only of a sum of squared terms instead of having a trigonometric term. From a topological point of view the two metrics are equivalent, but they use different scaling factors.

Preliminary simulations using the distance in the Riemannian metric show slightly better results in terms of true and false associations than the conventional approach with curvilinear coordinates. However, more

tests with, e.g., different orbits are necessary to assess the validity of the new distance definition, considering also the effects of the additional mean anomaly term and of the scaling in the Mahalanobis distance.

Furthermore, the results of simulations using the two different metrics will have to be compared to evaluate their applicability.

## 8 References

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