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Associating short-arc range and angle measurements of objects in LEO

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ABSTRACT

Observations of space debris objects in Low Earth Orbit (LEO) regime are usually conducted with radars. The observations may consist of sequences of range and angle measurements, so-called tracklets, usually covering only short arcs of the orbit.
The association of two or more sequences is necessary to calculate an initial orbit of the observed space object with sufficient accuracy. Orbits can be derived from observations in a single short arc and can be associated according to an orbital matching criterion.
In this work the association of more than two short-arc tracklets is studied. A multiple hypothesis approach is compared to a global approach based on a genetic algorithm. Different aspects related to initial orbit determination and orbital matching are also analysed. In particular, for the latter the Euclidean metric is applied to vectors in the orbital element space.

1 INTRODUCTION

Validating and improving space debris environment models requires regular monitoring of the space debris population. Today, statistically sufficient data is acquired by conducting optical surveys and by radar beam-park experiments, while novel observation techniques, such as laser means, are under development. Characterising the detected objects provides further input to both, the cataloguing and the modelling task. Such information may comprise the identification of progenitor objects of generated space debris, or, in particular for larger intact objects, the estimation of physical properties, such as, e.g., area-to-mass ratio, colour, tumbling state and attitude change rates. The data acquisition strategy implies that recent orbital data of the characterised objects are available, which are maintained in a database, i.e. in a catalogue, during the characterisation time frame.
The build-up of a catalogue and its maintenance depends on the capacity to determine the orbit of the observed objects from few measurements. In fact only a limited number of observations are available per night per object, each over observation arcs that can be as short as a few seconds. Therefore a single track, regardless of measurement type, often does not contain sufficient information in order to reliably estimate the observed object's state or conduct follow-up observations. For this reason the sparse observations or short sequences of observations (tracklets) need to be correlated or associated with each other.
Several approaches for optimal association or correlation have been described. The developed methods mainly relate to observations from optical sensors, as e.g. in [RD-1][RD-2][RD-3]. Nowadays several radar sensors are fully or partially devoted to space surveillance. Independent of light and weather conditions radar systems provide observations of space objects in low and medium earth orbits. There is less literature regarding the correlation of radar data. Gronchi et al. propose in [RD-4][RD-5] methods to correlate radar data based on the Keplerian integrals. A different approach is followed in [RD-14] where orbits from single tracklets are determined and then compared.
The tracklet association problem can be extended to three or more tracklets. This consideration brings us towards a Multiple Target Tracking (MTT) paradigm. The MTT problem is the natural extension of a pairwise tracklet correlation (two tracklets at two different epochs) to a correlation with three or more tracklets. A thorough discussion of the different approaches to solve this problem and the difficulties encountered can be found in [RD-6].
The only way to optimally solve the MTT problem is to try all the different combinations of observations and perform an orbit determination for each of these hypothetical objects. This quickly becomes computationally unfeasible. A popular and well known algorithm is the Multiple Hypothesis Tracking (MHT) algorithm. In [RD-7] a general explanation of the MHT principles is given. This algorithm enumerates all the possible combinations of tracklets and evaluates them. In order to keep the computational complexity reasonable the correlation is solved sequentially, over a sliding window. As a consequence the decision to correlate two tracklets can be postponed and taken later on the basis of additional measurements. This will decrease the number of false associations between tracklets.
A common approach to solve this type of combinatorial problems is to seek an approximation to the optimum solution, which can be found in a reasonable computation time. This kind of problems is solved in different branches adopting evolutionary algorithms. In the field of tracklet association an example of procedure using a genetic algorithm (GA) is given in [RD-8] and employs an orbit determination technique based on an optimized boundary value approach.

In this paper a method is proposed to associate multiple

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radar tracklets (range and angle measurements). For the pairwise association a scheme similar to the one proposed in [RD-14] and described in [RD-15] is used: calculation of initial orbit from radar tracklet, propagation of orbit to epoch of second tracklet, comparison of propagated orbit with orbit calculated from second radar tracklet (orbit matching), computation of the associated orbit. This scheme is applied for all the pairwise combinations of tracklets to be evaluated. However, two different approaches to the combinatorial problem are investigated. One based on a Multiple Hypothesis Tracking principle and the other on a genetic algorithm.
In addition to the combinatorial problem the orbit matching part is investigated. The association of the two orbits can be evaluated using the concept of distance between two orbits. The matching is then successful if the distance is smaller than a given threshold. In this work another definition of distance in the space of the orbital elements is used. In [RD-9] the author introduces a formula to compute the geodesic distance between points in the orbital element space. The results obtained with this distance definition are compared with the one obtained using the Mahalanobis distance in curvilinear coordinates.

2 MULTIPLE TRACKLET ASSOCIATION

2.1 Pairwise tracklet association

Two methods can be considered to compute the initial orbit: the Lambert method [RD-11], using two observations with angles and ranges, and the time difference between them, and the "Range and Angles method" described in the Goddard Trajectory Determination System (GTDS) document [RD-10], able to use more than two observations with an iteration scheme. The GTDS Range and Angles method provides in most of the tested cases a more accurate initial orbit determination than the Lambert method, probably because it can use all the observations of the tracklet.
Only the tracklets with a configured minimum number of observations are selected for the orbit determination process. The obtained initial orbit has still to be refined with a least squares approach where ranges and angles are weighted differently. At this stage, for some tracklets the least squares will fail to converge, while for other tracklets the root mean square (RMS) calculated in the least squares fitting exceeds a configured threshold and are discarded.
The orbit calculated from the radar tracklet is then propagated to the epoch of the second tracklet and is compared with the orbit of the second tracklet using the concept of distance between two orbits. Often the definition of Mahalanobis distance is used as a measure of the goodness of the association. The limitation with this measure is in the description of the uncertainty distribution, modeled according to the covariance in a Gaussian distribution. Mostly the Gaussian assumption is enough to describe the uncertainty in the orbital parameters, but depending on the coordinate system the inadequacy can be accentuated. A detailed explanation is given e.g. in [RD-13]. For example, in a Cartesian system is difficult to describe the typical "banana" shaped elongation of the error ellipsoid, due to the faster increase in the along-track uncertainty. Several methods have been developed to take into account non-Gaussian distributions in propagation and tracklet association. Sometimes an appropriate coordinate system can be found where the Gaussian assumption approximates the actual distribution. Curvilinear coordinates [RD-12][RD-13] are usually more suitable to describe the orbital uncertainty distribution. Essentially the transformation to curvilinear coordinates takes into account the real curved trajectory of the target. As a consequence in this coordinates system the expected "banana" shaped ellipsoid can be better approximated with a Gaussian distribution.
After the best tracklet association is found, the final orbit using the complete set of observations in the two associated tracklets is computed. Here a least squares improvement of the available radar initial orbit is performed. Different weights for radar and optical measurements may be adopted in the weight matrix.
The RMS obtained in the least squares fitting, and weighted according to the average measurement errors, is taken into account to still discard, defining a maximal value, the wrong tracklet associations.

2.2 Multiple tracklets

Two approaches are compared for the association of multiple tracklets. In the "direct" approach all possible combinations of tracklets are checked and if the combination satisfies the threshold requirements the association is accepted. For example, if there are 3 fences with 3 observed tracklets each, a total of 3 x 3 x 3 = 27 combinations is evaluated. This means that several solutions can be accepted in which the same tracklet belongs to different objects. To avoid this inconsistency a "global" approach needs to be used, where the solution as a whole is consistent. The price to pay is a much larger computation effort which usually can be handled only approximately e.g. by means of heuristic algorithms. In this work for the global approach a genetic algorithm was chosen. In both approaches, similarly to the Multiple Hypothesis Tracking algorithm, a sliding window defines the sets of tracklets (e.g. the number of fences) each to be considered in the association problem. The orbits calculated from the associations within the window are then used with new sets of tracklets in the next shifted position of the window.

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2.2.1 Genetic algorithm

2.2.1.1 Definition of individual

A genetic algorithm works with a population of individuals where each individual represents a potential solution and is evaluated to determine its so-called fitness, a measure of the quality of that solution. In our context an individual represents the associations between the tracklets. To describe an individual we recall the definition of k-matrix introduced in [RD-8].

In the k-matrix any entry k_{ij} can only have a value of 1 or 0. If $k_{ij}=1$ it signifies that the tracklet in row i is associated to the object in column j. The k-matrix is defined in such a way that the first tracklet is always associated to the first object. Following this logic the k-matrix becomes a lower triangular matrix. Besides this, each row may only contain one non-zero element.

2.2.1.2 Definition of fitness function

The definition of fitness function

$$f(k) = -\log P(k)$$

for the individual (or k-matrix) k depends on the probability

$$P(k) = \prod_i P(obj_i).$$

$P(obj_i)$ is the probability for associations of the single object obj_i (i.e. in the matrix column i). This probability has two components:

$$P(obj_i) = P_C(N) P_M.$$

The configuration component $P_C(N)$ is defined as

$$P_C(N) = (1-p_d)^{S-N} p_d^N (1-p_f)^N,$$

where p_d and p_f are the probability of true and false detection, N the number of tracklets for the object, S the maximal number of possible observed tracklets according to the observation strategy, e.g. number of fences. The orbit matching component P_M depends on the pairwise orbit distances d_{ij}:

$$P_M \sim \sum_{i<j} d_{ij}^2.$$

Note that this summation over the pairwise distances is used also within the sliding window in the direct approach. The sum is then divided by the number of combinations to obtain an averaged distance for the multiple association.

2.2.1.3 Definition of genetic algorithm

In a genetic algorithm the population of individuals changes from generation to generation. The individuals with a better fitness have a higher probability to pass on their information to the next generation. Several operators can be applied to the population at every generation.

The uniform crossover operator switches, with a certain probability, a row between two individuals, selected according to their relative fitness value, changing the object to which the tracklet is associated.

In the mutation operator each row can be mutated with a user defined mutation probability by randomly redefining the column where the '1' occurs, effectively assigning the tracklet to another object.

The two operators are applied until a new population is created. In the Elitist genetic algorithm the top few percent of the population is copied to the next generation, this ensures that the information contained within the best individuals is never lost.

The process of creating new generations is repeated until a maximum number of generations is reached.

The current settings of the algorithm, given a total of N tracklets, are: population size 2N, mutation probability 1/N, crossover probability 0.5, elitist fraction 10%, and maximal number of 150 generations.

2.3 Simulated measurements

Radar measurements of LEO objects on almost circular orbits (eccentricity < 0.01) at altitudes around 1000 km were simulated. The objects from the Space-Track TLE catalogue are observed during one night from a station at 40º latitude. Table 1 shows the values used for the simulation.

Radar pointing	Az. 180º, El. 60º
FoR	Az. 120º, El. 20º
Error (σ) in range	5 m
Error (σ) in angle	15'
Interval betw. obs.	10 s

Table 1. Values for the simulated radar observations for radar tracklets association.

The pairwise tracklet association procedure with the above described scheme was applied. The initial orbit was calculated with the GTDS method, propagated with a Keplerian model, and the Mahalanobis distance was computed in curvilinear coordinates. In the initial orbit determination only tracklets with at least 3 observations were considered. A threshold of 10 in the Mahalanobis distance and a threshold of 5 for the maximal acceptable RMS in the least squares calculation of initial and final associated orbit were chosen.

A total of 1024 tracklets is detected. After excluding too short tracklets and the tracklets where initial orbit determination fails, a net number of 193 tracklet pairs remains. Out of these, 100 tracklet triples were selected to test multiple tracklet correlation.

2.4 Results

Tests with the direct and global approach have been conducted with different numbers of objects to see the

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effect on the computation time. Sliding window sizes of 2 and 3 were chosen. Table 2 and Table 3 show the results for the direct and the global approach, respectively. True and false positives for the associated triples and the computation time are given. The colors indicate a good (green), medium (yellow) and bad (red) performance. The simulations were performed on a PC with 4 x Intel Core i5 processors at 3.20 GHz.

The best results in terms of true/false positives and computation time are obtained in the direct approach with window size 2. With size 3 not only the number of correct associations decreases but also the computation time gets worse up to 1 min for 100 objects. The change in terms of associations could be explained by the fact that the fitness function is an average over combinations of 3 tracklets and bad associations can be compensated. In the global approach there is in general a worse performance. For window size 2, the number of true positives is comparable with the direct approach case with size 3, but the number of false positives strongly increases. Besides this the computation time reaches several minutes for a relatively small number of objects. With window size 3 the situation still worsens with a smaller amount of correct associations and higher computing time. In the case with 100 objects the results were not evaluated since the simulation duration exceeded a reasonable time interval.

obj.	win = 2		win = 3	
	True / false	Time	True / false	Time
12	10 / 0	< 1 s	5 / 0	< 1 s
30	17 / 0	1 s	8 / 0	2 s
40	24 / 0	1 s	13 / 0	2 s
100	75 / 5	5 s	33 / 3	1 min

Table 2. Direct approach. Tracklet correlation with different numbers of objects and window sizes: true / false positives and computation time.

obj.	win=2		win=3	
	True/false	Time	True/false	Time
12	8 / 11	< 1 min	6 / 17	1 min
30	15 / 23	5 min	11 / 30	7 min
40	19 / 32	13 min	13 / 36	18 min
100	-	> 40 min	-	> 1h

Table 3. Global approach. Tracklet correlation with different numbers of objects and window sizes: true / false positives and computation time.

From the results of these preliminary investigations it seems that a direct approach with a sliding window is still a better solution for the multiple tracklet correlation problem. The direct approach allows us, through a selected threshold, to keep several possible solutions to be evaluated in the next window position. In the global method a single solution is kept for further combinations with new tracklets. It would be possible to introduce a threshold in order to keep several good global solutions, but this needs to be further investigated. But besides these considerations, by now the real limiting factor of the global approach seems to be the amount of time necessary to perform the association. In the scenarios with 100 objects already the generation of the initial population for the genetic algorithm, without any orbit evaluation, poses a problem and requires several minutes.

3 ORBIT MATCHING

3.1 Distance in orbit element space

We consider an alternative to the orbit matching through the Mahalanobis distance in curvilinear coordinates. In [RD-9] the space of bounded Keplerian orbits of fixed energy is described using the topology $V(E) \sim S^2 \times S^2$, the Cartesian product of two spheres. This topology can be extended through the semimajor axis to the cone $K(S^2 \times S^2)$ (see [RD-9] for more details). The formula to compute the geodesic distance between points in this space is:

$$d = \sqrt{2(a_1^2 + a_2^2 - 2a_1 a_2 \cos\Delta\psi} \qquad (1)$$

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where $$\Delta\psi = \sqrt{\frac{arccos^2(\vec{\eta}_1 \cdot \vec{\eta}_2) + arccos^2(\vec{\xi}_1 \cdot \vec{\xi}_2)}{2}} \quad (2)$$

and $$\vec{\eta} = \vec{e} + \vec{h}, \qquad \vec{\xi} = \vec{e} - \vec{h}. \quad (3)$$

Here is $\vec{e}$ the eccentricity vector and $\vec{h}$ a normalized angular momentum vector $\vec{h} = \frac{\vec{H}}{\sqrt{\mu a}}$ given the semimajor axis a and the gravitational parameter μ. The related Riemannian metric is induced by the Euclidean metric on $\mathbb{R}^6$. In the article it is mentioned that the geodesic distance on $K(S^1 \times S^1)$ can be generalized to a manifold with n spheres $K(S^1 \times \ldots \times S^1) \subset \mathbb{R}^{2n}$. The definition in (2) is then replaced by a general expression which contains not only specifically the angle differences on the sphere for $\vec{\eta}$ and $\vec{\xi}$, but additional angles for any additional sphere:

$$\Delta\psi = \sqrt{\frac{\sum_{i=1}^{n} \Delta\theta_i^2}{n}}. \quad (4)$$

We want to extend the distance between two orbits (1) to all 6 orbital parameters including the orbit anomaly. We assume we can describe the problem with a topology $K(S^2 \times S^2 \times S^1) \subset \mathbb{R}^8$ and the sphere S^1 should be related to the eccentric anomaly, according to the construction in [RD-16]. Then we have in (4) an additional angle difference $\Delta\theta_i$. For nearly circular orbits instead of the eccentric anomaly we consider the difference in true anomaly and we calculate this from the position vectors related to the two orbits at the osculating epoch. Using the metric given in [RD-9] there is one component g_{66} more in the metric tensor for the orbital elements $(x^i)_{i=1}^6 = (a, e, i, \Omega, \omega, \nu)$, with the following non-zero components:

$$g_{11} = 2$$

$$g_{22} = \frac{2a^2}{1 - e^2}$$

$$g_{33} = 2a^2(e^2 \sin^2 \omega + (1 - e^2))$$

$$g_{44} = 2a^2[e^2(1 - \sin^2 \omega \ \sin^2 i) + (1 - e^2) \ \sin^2 i]$$

$$g_{55} = 2a^2 e^2 (1 - \cos^2 \omega \ \sin^2 i)$$

$$g_{66} = a^2$$

$$g_{34} = 2a^2 e^2 (\sin \omega \ \cos \omega \ \sin i)$$

$$g_{45} = 2a^2 e^2 \cos i$$

In this case it is difficult to scale the distance depending on the covariance of the computed orbits, in a way similar to the case with the Mahalanobis distance. In the latter the inverse covariance matrix acts as a metric tensor scaling the coordinates system. So the above tensor should be modified to include the covariance information. In our tests no scaling through the covariance was applied. A threshold parameter for the distance was tuned to obtain best results.

3.2 Results

The correlation performance comparison using different distance definitions was conducted in terms of true and false positives using the data simulated in the first part. Table 4 shows the results using the Mahalanobis distance in Cartesian coordinates (as standard baseline), curvilinear coordinates, and the distance in the orbit element space. For these tests the RMS from the final orbit computation after the association is not used to discard wrong association. So the ratio true/false positives is only due to the threshold applied to the orbit matching. Also, for the comparison is useful to see the true positives for the same given number of false positives. For this purpose the threshold in the Cartesian and orbit elements case has been varied accordingly.

Cartesian coord.	Curvilinear coord.	Orbit elements
140 / 72	179 / 71	166 / 82

Table 4. True and false positives using the Mahalanobis distance in Cartesian and curvilinear coordinates, and the distance in the orbit element space.

The results obtained with curvilinear coordinates are slightly better than with the orbit element distance, which in turns performs better than using Cartesian coordinates. Table 5 shows the correlation results if also the RMS threshold is considered to accept or reject the association.

Cartesian coord.	Curvilinear coord.	Orbit elements
137 / 31	156 / 20	149 / 31

Table 5. True and false positives using the threshold in the distance and in the orbit RMS.

In general there is a slight decrease in the true positives and a massive reduction of false positives. This can be observed in the following diagrams. In Figure 1 the RMS distribution of false correlations is shown after a cut only with the Mahalanobis distance threshold. In Figure 2 the distance distribution after only an RMS cutoff is shown. We see that one cutoff criterion alone is not enough, and only the combination of both effectively removes a significant amount of false positives as illustrated in Figure 3.

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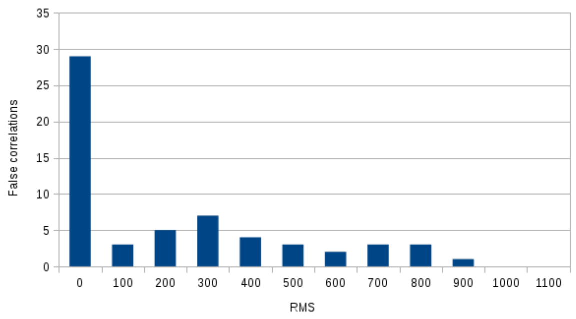

Figure 1: False correlation RMS distribution after only Mahalanobis distance threshold cut.

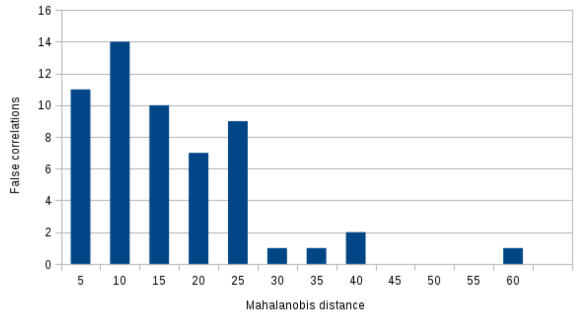

Figure 2: False correlation Mahalanobis distance distribution after only RMS threshold cut.

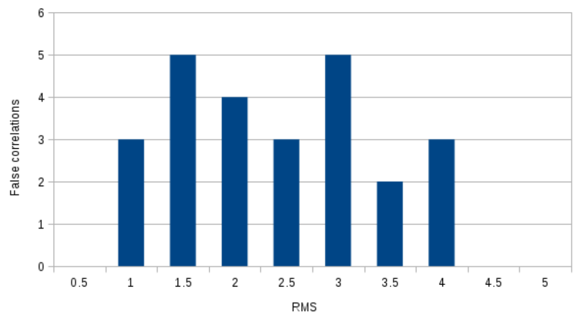

Figure 3: False correlation RMS distribution after both Mahalanobis distance and RMS threshold cut.

4 CONCLUSIONS

A comparison of two methods to associate multiple radar tracklets is presented. A direct approach, similar to the Multiple Hypothesis Tracking algorithm, and a global approach, using a genetic algorithm, are proposed. Both approaches base on a common procedure for the pairwise tracklet correlation which consists in different steps: the initial orbit determination from a single radar tracklet, the propagation to the epoch of the second tracklet, the comparison of the propagated orbit with an orbit calculated from the second radar tracklet, and the computation of the associated orbit. In the tested scenario and with the given definitions of fitness function and orbit matching, the results show a better association with the direct approach and a sliding window size of 2. The simulations with the genetic algorithm show a clear limitation in the computation time which exceeds several minutes even for scenarios with only 30 objects.

Two different distance definitions for orbit matching were compared regarding the correlation performance. The alternative to the commonly used Mahalanobis distance is a definition of distance in the space of orbital elements. This space can be described by a different topology and the geodesic distance between two points can be calculated in the defined manifold. The results show that using the alternative distance, values comparable to the case with the Mahalanobis distance in curvilinear coordinates are obtained. The scaling of this distance according to the orbit covariance has still to be implemented. Besides the importance of the distance definition, it is shown that a second criterion for orbit matching based on an RMS threshold is necessary to effectively reduce false associations.

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